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Non-Gaussian Scaling Limits. Hierarchical Model Approximation

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We extend a previous analysis of non-Gaussian fixed points of the block spin renormalization group in hierarchical models to a long distance analysis of the correlation functions.

KEY WORDS: Non-Gaussian critical point; block spin renormalization group; fixed point; hierarchical model; $1/N$ expansion; expansion.

1. INTRODUCTION

The present paper completes the analysis[1] of non-Gaussian fixed points of the renormalization group (RG) in the context of hierarchical models. In Ref. 1 such fixed point Gibbs factors were shown to exist in the case of small ϵ for model with long-range interactions decaying as $1/|x-y|^{d/2-\epsilon}$ or large N, for an N-component spin system. Here we extend this analysis by showing that the long distance behavior is, indeed, governed by the non-Gaussian fixed point. To recall from Ref. 1 (which the reader should have in hand), the Gibbs states we consider are given, in a box Λ_M of volume L^{Md} in $\mathbb{Z}^d$, by

$$\langle - \rangle_{v_0} = \frac{1}{\mathsf{N}} \int (-) \exp\Big[- \sum_{x \in \Lambda_M} v_0(\phi_x) \Big] d\mu_G(\phi) \equiv \int (-)\, d\rho(\phi) \qquad (1)$$

where $d\mu_G$ is a Gaussian measure with covariance G having the decay

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$1/|x-y|^{\alpha}$ as $|x-y| \to \infty$ (at $M=\infty$). In the ϵ case $\alpha = d/2 - \epsilon$ and in the $1/N$ case $\alpha = d-2$. $\phi_x = \phi_{xi}$, $i = 1, \ldots, N$, $N = 1$ in the ϵ case. The explicit form of G is not relevant here; we only need the following facts about the RG, defined as map between Gibbs states. Namely, the transformed state,

$$\langle - \rangle' = \int (-)\, d\rho'(\phi) \tag{2}$$

originally given by the block spin integral

$$d\rho'(\phi) = \frac{1}{\mathsf{N}_1} \int \delta(\phi' - C\phi)\, d\rho(\phi) \tag{3}$$

$$C\phi_x = L^{\alpha/2-d} \sum_{|y_\mu| < L/2} \phi_{Lx+y} \tag{4}$$

is in our hierarchical model given by

$$d\rho'(\phi) = \frac{1}{\mathsf{N}'} \exp\left[- \sum_{x \in \Lambda_{M-1}} v_1(\phi_x) \right] d\mu_G(\phi) \tag{5}$$

(G and ϕ are now on Λ_{M-1}) with

$$\exp[-v_1(\phi)] = \int \exp[-w_0(\phi, \mathbf{z})]\, d\nu(\mathbf{z}) \tag{6}$$

$$w_0(\phi, z) = \frac{L^d}{2}\left(v_0(\psi^+) + v_0(\psi^-)\right) \tag{7}$$

$$\psi^+ = L^{-(\alpha/2)}\psi \pm \mathbf{z} \tag{8}$$

$$d\nu(\mathbf{z}) = \frac{1}{(2\pi)^{N/2}} e^{-(1/2\mathbf{z}^2)}\, d\mathbf{z} \tag{9}$$

The transformation $e^{-v_0} \to e^{-v_1}$ was in Ref. 1 shown to have a non-Gaussian fixed point for ϵ small, $N=1$ or $\alpha = d-2$, N large, and the stable manifold in the vicinity of the fixed point was also constructed. Here we wish to show that the states indeed exist in the $M \to \infty$ limit and are critical with α being the exponent determining the infrared behavior of the correlations. That in the ϵ case the Gaussian (unperturbed) and non-Gaussian theories have the same exponent is an artifact of the hierarchical model. In constrast, we will show, that in the scaling limit these theories indeed are non-Gaussian, governed by the non-Gaussian fixed point: the truncated correlations will not vanish. The analysis of the correlations below parallels the one for the Gaussian fixed point.[2] There is, however, an extra difficulty due to the fact that the so-called "small field region" does not expand in the process of iteration of the RG, owing to the non-Gaussian part of the Hamiltonian which does not contract away. This problem is resolved by a more careful large field analysis.

2. THE SMALL ϵ CASE

We take now $\alpha = d/2 - \epsilon$ with ϵ small and $N = 1$. For definiteness take also $d = 3$. Consider an arbitrary correlation

$$\langle \phi_{x_1} \cdots \phi_{x_n} \rangle_{v_0} \tag{1}$$

As recalled from Refs. 1 and 2, the successive block spin decompositions correspond to expanding the field ϕ as a sum of fluctuations

$$\phi_x = \sum_{n=0}^{M-1} L^{-(\alpha n/2)} \mathsf{A}(x_n) z_{n x_{n+1}} \qquad \left(x_n = \left[\frac{x}{L^n} \right] \right) \tag{2}$$

(here A is a fixed function taking values ± 1). We evaluate (1) by successively integrating out the z fields. Each such integration factors over blocks as in the case of the Gibbs factors. Denoting

$$\langle - \rangle_n = \int (-) \exp[-w_n(\phi, z)] \, d\nu(z) \Big/ \int \exp[-w_n(\phi, z)] \, d\nu(z) \tag{3}$$

we get

$$\left\langle \prod_i \phi_{x_i} \right\rangle_{v_0} = \left\langle \prod_{x \in \Lambda_{M-1}} \left\langle \prod_{[x_i/L] = x} \phi_{x_i} \right\rangle_0 \right\rangle_{v_1} \tag{4}$$

$$\equiv \left\langle \prod_x G_x^1 \right\rangle_{v_1} = \left\langle \prod_y \left\langle \prod_{[x/L]=y} G_x^1 \right\rangle_1 \right\rangle_{v_2}$$

$$\equiv \left\langle \prod_x G_x^2 \right\rangle_{v_2} = \cdots = \left\langle \prod_x G_x^n \right\rangle_{v_n} \tag{5}$$

After a finite number m of iterations, namely, as $x = [x_i/L^m] = [x_j/L^m]$ for all i, j, we have only one $G_x^n \neq 1$ in (5). Then we iterate

$$\langle G_x^m \rangle_{v_m} = \langle \langle G_x^m \rangle_m \rangle_{v_{m+1}} \equiv \langle G^{m+1} \rangle_{v_{m+1}} = G^M(0) \tag{6}$$

We thus wish to control the operation

$$G \to G' = \langle G \rangle_m \tag{7}$$

Before tackling this, let us recall the Gibbs factors attracted by the fixed point. We considered $\exp[-v_0]$ given by

$$\exp[-v_0] \equiv g_0 = \tilde{g}_0 e^{-a_0 \phi^2} \tag{8}$$

and satisfying the following:

(a) $\tilde{g}_0$ is even, analytic in $|\mathrm{Im}\,\phi| < |\log \epsilon|$, $\tilde{g}(0) = 1$, $\tilde{g}''(0) = 0$.

(b) For $|\phi| < |\log \epsilon|$ $\tilde{v}_0(\phi) = -\log \tilde{g}_0(\phi)$ is analytic with

$$\tilde{v}_0(\phi) = \lambda_0 \phi^4 + \tilde{\tilde{v}}_0(\phi), \qquad \tilde{\tilde{v}}_0(0) = \tilde{\tilde{v}}_0''(0) = \tilde{\tilde{v}}_0''''(0) = 0$$

$$|\lambda_0 - \bar{\lambda}| \leqslant \epsilon^{3/2}, \qquad \sup_{|\phi| < |\log \epsilon|} |\tilde{\tilde{v}}_0| \leqslant \epsilon^{7/4}$$

with $\bar{\lambda} = (1/36) L^{-3}(1 - L^{2\alpha - 3})$.

(c) If $|\phi| \geqslant |\log \epsilon|$, $|\mathrm{Im}\,\phi| < L^{-\alpha/2}|\log \epsilon|$ then

$$|\tilde{g}_0(\phi)| \leqslant \exp\left[-\frac{\bar{\lambda}}{4}\left((\mathrm{Re}\,\phi)^4 + \kappa(\mathrm{Re}\,\phi)^2\right)\right]$$

Then it was shown that there exists $a_0 = a_{\mathrm{crit}}$ in $[-3\epsilon^{2/3}, 3\epsilon^{2/3}]$ such that g_n, the nth renormalized g, converges to a fixed point g^*, g_n satisfying (a)–(c) too, with $a_n \in [-3\epsilon^{2/3}, 3\epsilon^{2/3}]$. Since we wish to show that the scaling limit is controlled by this g^*, let us sharpen the analysis of Ref. 1 slightly. Namely, one can easily strengthen the knowledge on a_{crit} and g^*.

Let us first apply the analysis of Ref. 1 to the fixed point $g^* = \tilde{g}^* e^{-a^*\phi^2}$ which was shown to exist. The estimates of Ref. 1: (2.19)–(2.21) together with (2.37) give

$$\lambda^* = L^{2\epsilon}\lambda^* - 36L^{6-3\alpha}\lambda^{*2} + O(\epsilon^{8/3})$$

$$\Rightarrow \lambda^* = \frac{\epsilon \log L}{18L^{3/2}} + O(\epsilon^{5/3}) \tag{9}$$

$$a^* = L^{3-\alpha}a^* - 6L^{3-\alpha}\lambda^* + O(\epsilon^{5/3})$$

$$\Rightarrow a^* = \frac{6\lambda^*}{1 - L^{-3/2}} + O(\epsilon^{5/3}) \tag{10}$$

and then iterated once more, now with the *a priori* estimate $a^* = O(\epsilon)$:

$$\lambda^* = \frac{\epsilon \log L}{18L^{3/2}} + O(\epsilon^2), \qquad a^* = \frac{6\lambda^*}{1 - L^{-3/2}} + O(\epsilon^2) \tag{11}$$

Now consider iterating (a)–(c) as in Ref. 1. Especially, look at a_1. Let us assume, instead of $a_0 \in [-3\epsilon^{2/3}, 3\epsilon^{2/3}]$ that

$$|a_0 - a^*| < \epsilon^{4/3} \tag{12}$$

Then from Ref. 1: (2.19) and (2.37) [compare with Ref. 1: (3.2)]

$$a_1 = L^{3/2}a_0 + 6L^{3/2}\lambda_0 + O(\epsilon^2) \tag{13}$$

whence [using (9) and (b)]

$$a_1 - a^* = L^{3/2}(a_0 - a^*) + (L^{3/2} - 1)a^* + 6L^{3/2}\lambda^* + 6L^{3/2}(\lambda_0 - \lambda^*) + O(\epsilon^2) = L^{3/2}(a_0 - a^*) + O(\epsilon^{3/2}) \tag{14}$$

Hence there is a closed interval $I_1 \subset I_0 = [a_* - \epsilon^{4/3}, a_* + \epsilon^{4/3}]$ such that as a_0 sweeps I_1, a_1 sweeps I_0. Thus (a)–(c) and (12) are stable with iteration.

Of course $a_n \to a^*$ too. So, what we need to know of g_n is that

$$|a_n - a^*| \leqslant \epsilon^{4/3}, \qquad |\lambda_n - \lambda^*| \leqslant 2\epsilon^{3/2} \tag{15}$$

$$a_n = \frac{6\lambda_n}{1 - L^{-3/2}} + O(\epsilon^2) \tag{16}$$

and that (a)–(c) hold.

Let us now show, that in the scaling limit the model defined by (8) and (a)–(c) indeed is non-Gaussian. We will, for simplicity, consider the connected four-point function

$$G_4^c(x_1 \ldots x_4) = \langle \phi_{x_1} \ldots \phi_{x_4} \rangle_{v_0}^T = \langle \phi_{x_1} \ldots \phi_{x_4} \rangle_{v_0} - \sum_{\Pi} \langle \phi_{x_i} \phi_{x_j} \rangle_{v_0} \langle \phi_{x_k} \phi_{x_l} \rangle_{v_0} \tag{17}$$

We wish to show, that (at $M = \infty$)

$$\lim_{k \to \infty} L^{2k\alpha} |G_4^c(L^k x_1, \ldots, L^k x_4)| \tag{18}$$

exists and is nonvanishing (the reason we take absolute values is that due to the nontranslational invariance of the hierarchical model, the scaling limit as such is not well defined).

To see the argument clearly, let us pick the points $x_1 \ldots x_4$ such that $[x_i / L^n] \equiv x_{in}$, as n increases, simultaneously fall to the same block: let n_0 be the first integer such that $x_{in_0} = x_{jn_0}$ for all i, j. We have to compute separately the two-point and four-point functions. As in (4) and (5)

$$\begin{aligned}
&\langle \phi_{L^k x_1} \ldots \phi_{L^k x_4} \rangle_{v_0} \\
&= L^{-2\alpha} \langle \phi_{L^{k-1} x_1} \ldots \phi_{L^{k-1} x_4} \rangle_{v_1} \\
&= L^{-2\alpha(n_0 + k - 1)} \langle \phi_{x_{1 n_0 - 1}} \ldots \phi_{4_{n_0 - 1}} \rangle_{v_{n_0 + k - 1}} \\
&= L^{-2\alpha(n_0 + k)} \langle \phi^4_{x_{1 n_0}} \rangle_{v_{n_0 + k}} \\
&\quad + L^{-2\alpha(n_0 + k - 1)} \mathsf{A}(x_{1 n_0 - 1}) \ldots \mathsf{A}(x_{4 n_0 - 1}) \langle \langle z^4_{n_0 + k - 1} \rangle_{n_0 + k - 1} \rangle_{v_{n_0 + k}} \\
&\quad + L^{-2\alpha(n_0 + k - 1)} L^{-\alpha} \langle \phi^2_{x_{1 n_0}} \langle z^2_{n_0 + k - 1} \rangle_{n_0 + k - 1} \rangle_{v_{n_0 + k}} \sum_{\text{pairs}} \mathsf{A}(x_{i n_0 - 1}) \mathsf{A}(x_{j n_0 - 1}) \\
&= \sum_{m = n_0 + k - 1}^{M - 1} \Bigg\langle \Bigg[A_m L^{-2\alpha m} \langle z_m^4 \rangle_m \\
&\qquad + \sum_{n_0 + k - 1}^{m - 1} B_l L^{-2\alpha l} L^{-\alpha(m - l)} \langle z_m^2 G_m^l \rangle_m \Bigg] \Bigg\rangle_{v_{m+1}}
\end{aligned} \tag{19}$$

where we denoted

$$A_{n+k-1} = \prod_{i=1}^{4} \mathsf{A}(x_{in_0-1}), \qquad A_m = 1 \quad \text{otherwise}$$
$$B_{n+k-1} = \sum_{\text{pairs}} \mathsf{A}(x_{in_0-1})\mathsf{A}(x_{jn_0-1}), \qquad B_m = 3 \quad \text{otherwise} \tag{20}$$

and

$$G_m^l = \langle G_{m-1}^l \rangle_{m-1}, \qquad G_{l+1}^l = \langle z_l^2 \rangle_l \tag{21}$$

Compute $\langle\langle z_m^4 \rangle_m \rangle_{v_{m-1}}$ now. Let

$$F_{m+1} = g_{m+1}\langle z_m^4 \rangle_m = \int z_m^4 f_m(\phi, z_m)\, dv(z_m) \Big/ \int g_m(z_m)^{L^3} dv(z_m) \tag{22}$$

with

$$f_m(\phi, z) = g_m(\psi^+)^{L^3/2} g_m(\psi^-)^{L^3/2} \tag{23}$$

$$F_{n+1} = \int F_n(\psi^+) f_n(\phi, z_n) g_n(\psi^+)^{-1} dv \Big/ \int g_n(z)^{L^3} dv \tag{24}$$

Then

$$\langle\langle z_m^4 \rangle_m \rangle_{v_{m+1}} = F_M(0) \tag{25}$$

Let us study F_{m+1}. For $|\phi| < |\log \epsilon|$, we write ($\chi = \chi(|z| < \delta |\log \epsilon|)$; see Ref. 1:

$$\begin{aligned} F_{m+1} &= g_{m+1}\Big(\int z_m^4 e^{-w_m}\chi\, dv + \int z_m^4 f_m \chi^{\perp} dv\Big) \Big/ \Big(\int e^{-w_m}\chi\, dv + \int f_m \chi^{\perp} dv\Big) \\ &= g_{m+1}\langle z_m^4 \rangle_m^{\chi}(1 + r_m) \end{aligned} \tag{26}$$

with r_m analytic and bounded by

$$|r_m| < \epsilon^{O(|\log \epsilon|)} \tag{27}$$

Taylor expanding, we obtain

$$\begin{aligned} &(1 + r_m)\langle z_m^4 \rangle_m^{\chi} \\ &\quad \equiv (1 + r_m)\int z_m^4 \exp\big[+ L^3 a_m z^2 - L^{3-\alpha}\lambda_m \phi^2 z^2 - L^3 \lambda_m z^4 \\ &\qquad\qquad - L^3 \tilde{w}_m(\phi, z)\big]\chi\, dv \Big/ \int \exp\big[L^3 a_m z^2 - \cdots \big]\chi\, dv \\ &\quad = \alpha_{m+1} + \beta_{m+1}\phi^2 + \tilde{\tilde{G}}_{m+1}(\phi) \end{aligned} \tag{28}$$

with

$$\alpha_{m+1} = 3 + 12L^3 a_m - 96L^3\lambda_m + O(\epsilon^2) \tag{29}$$

$$\beta_{m+1} = -12L^{3/2}\lambda_m + O(\epsilon^2) \tag{30}$$

$$|\tilde{\tilde{G}}_{m+1}(\phi)| \leqslant O(\epsilon^2) \tag{31}$$

For the large field, $|\phi| \geqslant |\log\epsilon|$, $|\mathrm{Im}\,\phi| < L^{-\alpha/2}|\log\epsilon|$, we write

$$F_{m+1} = \alpha_{m+1} g_{m+1} + \tilde{F}_{m+1} \tag{32}$$

and have (there is considerable freedom here)

$$|\tilde{F}_{m+1}(\phi)| \leqslant (\mathrm{Re}\,\phi)^4 \exp\left[-\frac{\bar{\lambda}}{4}\left((\mathrm{Re}\,\phi)^4 + \kappa(\mathrm{Re}\,\phi)^2\right)\right] \tag{33}$$

We now claim iteratively: (A) F_n is analytic in $|\mathrm{Im}\,\phi| < |\log\epsilon|$, even and

$$F_n = \alpha_n g_n + \tilde{F}_n, \qquad \tilde{F}_n(0) = 0$$

(B) For $|\phi| < |\log\epsilon|$

$$\tilde{F}_n = g_n \tilde{G}_n, \qquad \tilde{G}_n = \beta_n\phi^2 + \tilde{\tilde{G}}_n(\phi)$$

with

$$|\alpha_n - \alpha_{n-1} - \beta_{n-1}| \leqslant d_1 L^{-\alpha(n-m)}\epsilon^2$$

$$|L^\alpha\beta_n - \beta_{n-1}| \leqslant d_2 L^{-\alpha(n-m)}\epsilon^2$$

$$|\tilde{\tilde{G}}_n(\phi)| \leqslant d_3 L^{-\alpha(n-m)}\epsilon^2, \qquad \tilde{\tilde{G}}_n^{(i)}(\phi)|_{\phi=0} = 0, \qquad i = 1, \ldots, 3$$

(C) For $|\phi| \geqslant |\log\epsilon|$, $|Im\,\phi| < L^{-\alpha/2}|\log\epsilon|$,

$$|\tilde{F}_n| \leqslant L^{-\alpha(n-m)}(\mathrm{Re}\,\phi)^4 \exp\left[-\frac{\bar{\lambda}}{4}\left((\mathrm{Re}\,\phi)^4 + \frac{\kappa}{2}(\mathrm{Re}\,\phi)^2\right)\right]$$

The iteration of (A)–(C) is straightforward: Consider (B). We get, as in (26),

$$F_{n+1} = \alpha_n g_{n+1} + \tilde{G}'_{n+1} g_{n+1} \tag{34}$$

with

$$\begin{aligned}\tilde{G}'_{n+1} &= \langle \tilde{G}_n \rangle_n^\chi \left(1 + \frac{\int f_n \chi^\perp\, dv}{\int e^{-w_n}\chi\, dv}\right)^{-1} \\ &\quad + \int \tilde{F}_n g_n (\psi^+)^{-1} f_n \chi^\perp\, dv \Big/ \int f_n\, dv \\ &= \langle \tilde{G}_n \rangle_n^\chi \left(1 + O(\epsilon^{O(|\log\epsilon|)})\right) + O(L^{-\alpha(n-m)}\epsilon^{O(|\log\epsilon|)})\end{aligned} \tag{35}$$

and all the factors analytic in fact in $|\phi| < (1-\delta)L^{\alpha/2}|\log\epsilon|$. Taylor expanding

$$\begin{aligned}\langle\tilde{G}_n\rangle_n^\chi &= L^{-\alpha}\beta_n\phi^2 + \beta_n\langle z^2\rangle_n^\chi + \langle\tilde{\tilde{G}}_n\rangle^\chi \\ &= \beta_n\langle z^2\rangle_n^\chi|_{\phi=0} + \langle\tilde{\tilde{G}}_n\rangle_n^\chi|_{\phi=0} \\ &\quad + \phi^2\left(L^{-\alpha}\beta_n + \frac{1}{2}\frac{d^2}{d\phi^2}\bigg|_{\phi=0}\langle\beta_n z^2 + \tilde{\tilde{G}}_n\rangle_n^\chi\right) \\ &\quad + \tilde{\tilde{G}}_n'(\phi) \end{aligned} \tag{36}$$

we estimate

$$\langle z^2\rangle_n^\chi|_{\phi=0} = 1 + O(\epsilon) \tag{37}$$

$$\left|\langle\tilde{\tilde{G}}_n\rangle_n^\chi|_{\phi=0}\right|, \qquad \left|\frac{d^2}{d\phi^2}\langle\tilde{\tilde{G}}_n\rangle_n^\chi|_{\phi=0}\right| \leqslant Cd_3L^{-\alpha(n-m)}\epsilon^2 \tag{38}$$

$$\left|\frac{d^2}{d\phi^2}|_{\phi=0}\langle z^2\rangle_n^\chi\right| \leqslant C\epsilon \tag{39}$$

$$|\tilde{\tilde{G}}_n'(\phi)| \leqslant \frac{1}{2}d_3L^{-\alpha(n+1-m)}\epsilon^2, \qquad \text{for} \quad |\phi| < |\log\epsilon| \tag{40}$$

where for $\tilde{\tilde{G}}_n'$ we got contraction due to restriction to a smaller region (see Ref. 1). (34)–(40) yield (B) for $n+1$.

As for (C), note that

$$\begin{aligned}\tilde{F}_{n+1} &= F_{n+1} - \alpha_{n+1}g_{n+1} \\ &= (\alpha_n - \alpha_{n+1})g_{n+1} + \int\tilde{F}_n(\psi^+)g_n(\psi^+)^{-1}f_n\,dv \Big/ \int g(z)^{L^3}dv(z) \end{aligned} \tag{41}$$

As in Ref. 1 [see (2.38) on], we consider separately $|z| < \frac{1}{2}L^{-\alpha/2}|\mathrm{Re}\,\phi|$ and $|z| \geqslant \frac{1}{2}L^{-\alpha/2}|\mathrm{Re}\,\phi|$. Note that (C) holds for all ϕ provided we replace $(\mathrm{Re}\,\phi)^4$ by $(\mathrm{Re}\,\phi)^4 + 1$. Thus for $|z| < \frac{1}{2}L^{-\alpha/2}|\mathrm{Re}\,\phi|$

$$(\mathrm{Re}\,\psi^+)^4 + 1 \leqslant \tfrac{1}{4}L^{-\alpha}(\mathrm{Re}\,\phi)^4 \tag{42}$$

(since $|\mathrm{Re}\,\phi| > |\log\epsilon|$), whereas for $|z| \geqslant \frac{1}{2}L^{-\alpha/2}|\mathrm{Re}\,\phi|$

$$\begin{aligned}(\mathrm{Re}\,\psi^+)^4 + 1 &\leqslant 4\big(L^{-2\alpha}(\mathrm{Re}\,\phi)^4 + z^4\big) + 1 \\ &< L^{-\alpha}(\mathrm{Re}\,\phi)^4(1 + z^4)\end{aligned} \tag{43}$$

The analysis of Ref. 1 may now be applied to the second term in (41) giving for it a bound

$$L^{-\alpha(n-m)}\tfrac{1}{2}L^{-\alpha}(\mathrm{Re}\,\phi)^4\exp\Big[-(\bar{\lambda}/4)(\ldots)\Big]. \tag{44}$$

Owing to (B) and (C) for g_n [recall (8) and $a_n = O(\epsilon)$], we now get (C) for $n+1$.

As a result, from (25) (taking also $M \to \infty$)

$$\langle\langle z_m^4 \rangle_m \rangle_{v_{m+1}} = 3 + 12L^3 a_m - 96L^3\lambda_m + \sum_{l=0}^{\infty} L^{-\alpha l}\beta_{m+1+l} + O(\epsilon^2)$$

$$= 3 + 12L^3\left(a_m - 8\lambda_m - \frac{L^{-3/2}\lambda_m}{1 - L^{-3/2}} \right) + O(\epsilon^2) \qquad (45)$$

Next consider the G_m^l of (21). Write again

$$g_m G_m^l \equiv F_m^l \qquad (46)$$

Then a similar analysis as above establishes (A)–(C) for F_m^l, the only difference being the starting values (29) and (30), which now are

$$\alpha_{l+1}^l = 1 + 2L^3 a_l - 12L^3\lambda_l + O(\epsilon^2) \qquad (47)$$

$$\beta_{l+1}^l = -2L^{3/2}\lambda_l + O(\epsilon^2) \qquad (48)$$

Hence for $|\phi| < |\log \epsilon|$

$$G_m^l = \alpha_m^l + \beta_m^l \phi^2 + \tilde{\tilde{G}}_m^l(\phi) \qquad (49)$$

with

$$\alpha_m^l = 1 + 2L^3 a_l - 12L^3\lambda_l - 2L^{3/2}\lambda_l \frac{1 - L^{-3/2(m-l)}}{1 - L^{-3/2}} + O(\epsilon^2) \qquad (50)$$

$$\beta_m^l = L^{-\alpha(m-l-1)}\left(-2L^{3/2}\lambda_l + O(\epsilon^2)\right) \qquad (51)$$

$$|\tilde{\tilde{G}}_m^l| < dL^{-\alpha(m-l)}\epsilon^2 \qquad (52)$$

Now we may compute $\langle\langle z_m^2 G_m \rangle_m \rangle_{v_{m+1}}$ in (19), again setting up an iteration as above. We get for

$$g_{m+1}\langle z_m^2 G_m^l \rangle_m \equiv \bar{F}_{m+1}^l \qquad (53)$$

(A)–(C) [with some multiplicative factor in (C)] and from (49)–(52), for $|\phi| < |\log \epsilon|$,

$$\bar{F}_{m+1}^l = g_{m+1}\left(\bar{\alpha}_{m+1}^l + \bar{\beta}_{m+1}^l \phi^2 + \tilde{\tilde{G}}_{m+1}^l(\phi)\right) \qquad (54)$$

with

$$\bar{\alpha}_{m+1}^l = \alpha_{m+1}^l \alpha_{m+1}^m - 2\beta_m^l + O(\epsilon^2) \qquad (55)$$

$$\bar{\beta}_{m+1}^l = \left(\alpha_{m+1}^m \beta_{m+1}^l + \alpha_{m+1}^l \beta_{m+1}^m\right) + O(\epsilon^2 L^{-\alpha(m-l)}) \qquad (56)$$

Thus it follows that

$$\langle\langle z_m^2 G_m^l \rangle_m \rangle_{v_{m+1}} = \langle\langle z_m^2 \rangle\rangle_{v_{m+1}} \langle\langle z^2 \rangle\rangle_{v_{l+1}} - 2\beta_m^l + O(\epsilon^2) \qquad (57)$$

with

$$\langle\langle z_m^2\rangle_m\rangle_{v_{m+1}} = 1 + 2L^3(a_m - 6\lambda_m) - \frac{2L^{3/2}\lambda_m}{1 - L^{-3/2}} + O(\epsilon^2)$$

$$= 1 + \frac{10L^{3/2}\lambda_m}{1 - L^{-3/2}} + O(\epsilon^2) \tag{58}$$

where we recalled that

$$a_m = \frac{6\lambda_m}{1 - L^{-3/2}} + O(\epsilon^2) \tag{59}$$

Since the two-point function is given by

$$\langle\phi_{L^k x_i}\phi_{L^k x_j}\rangle_{v_0} = \sum_{m=n_0+k}^{M-1} L^{-\alpha m}\langle\langle z_m^2\rangle_m\rangle_{v_{m+1}} + \mathsf{A}(x_{in_0-1})\mathsf{A}(x_{jn_0-1})$$
$$\times \langle\langle z_{n_0+k-1}^2\rangle_{n_0+k-1}\rangle_{v_{n_0+k}} \tag{60}$$

we may combine (19), (57), and (60) to get (at $M = \infty$)

$$\langle\phi_{L^k x_1}\cdots\phi_{L^k x_4}\rangle_{v_0}^T$$
$$= \sum_{m=n_0+k-1}^{\infty}\Bigg[A_m L^{-2\alpha m}\left(\langle\langle z_m^4\rangle_m\rangle_{v_{m+1}} - 3\langle\langle z_m^2\rangle_m\rangle_{v_{m+1}}^2\right)$$
$$-2\sum_{l=n_0+k-1}^{m-1} B_l L^{-\alpha(m+l)}\beta_m^l\Bigg] + L^{-2\alpha(n_0+k)}O(\epsilon^2) \tag{61}$$

By (45), (58), and (59)

$$\langle\langle z_m^4\rangle_m\rangle_{v_{m+1}} - 3\langle\langle z_m^2\rangle_m\rangle_{v_{m+1}}^2 = -24\lambda_m L^3\left(1 + O(L^{-3/2})\right) + O(\epsilon^2) \tag{62}$$

Recalling that (see Ref. 1)

$$|\lambda_m - \lambda_{m+1}| \leqslant (1-\epsilon)^m\epsilon^{3/2} \tag{63}$$

whence for k large enough

$$|\lambda_{n+k} - \lambda^*| \leqslant O(\epsilon^2) \tag{64}$$

we obtain from (61), (51), and (62)

$$\lim_{k\to\infty} L^{2\alpha k}|\langle\phi_{L^k x_1}\cdots\phi_{L^k x_4}\rangle_{v_0}^T| = +24L^{-2\alpha n}\lambda^*\left(1 + O(L^{-3/2})\right)L^6 + O(\epsilon^2)$$

(recall that $|\mathsf{A}| = 1$), i.e., the scaling limit is non-Gaussian, determined by the fixed point (it is indeed easy to show that as $k\to\infty$ only the fixed point contributes).

The other correlation functions are bounded in a similar way (see also Ref. 2). Above we got as a by-product for the two-point function, from (60) and (58), that

$$\langle \phi_x \phi_y \rangle_{v_0} = \sum_{m=n}^{\infty} L^{-\alpha m}\left(1 + \frac{10L^{3/2}\lambda_m}{1 - L^{-3/2}}\right)$$
$$+ \left[\mathsf{A}(x_{n-1})\mathsf{A}(y_{n-1})\left(1 + \frac{10L^{3/2}\lambda_{n-1}}{1 - L^{-3/2}}\right) + O(\epsilon^2)\right] L^{-\alpha n}$$

and thus the decay $1/|x-y|^\alpha$ as $|x-y| \to \infty$.

For such correlations the decay with exponent α comes somewhat trivially, since $\langle \phi \rangle_m = L^{-\alpha/2}\phi$. For completeness, let us show, that this decay persists for more complicated correlations too. Consider a two-point function

$$\langle \phi_x^m ; \phi_y^m \rangle_{v_0}^T = \begin{cases} \langle \phi_x^m \phi_y^m \rangle_{v_0}, & m \text{ odd} \\ \langle \phi_x^m \phi_y^m \rangle_{v_0} - \langle \phi_x^m \rangle_{v_0} \langle \phi_y^m \rangle_{v_0}, & m \text{ even} \end{cases} \tag{65}$$

Let m be odd first. Let again

$$F_1 = \int (\psi^+)^m f_0 \, dv \Big/ \int g(z)^{L^3} dv \tag{66}$$

and F_n be as before. We show inductively the following:

1. F_n is analytic in $|\mathrm{Im}\,\phi| < |\log \epsilon|$ with
$$F_n = \alpha_n g_n \phi_{x_n} + \tilde{F}_n; \qquad \tilde{F}_n'(0) = 0$$
2. For $|\phi| < |\log \epsilon|$, $\tilde{F}_n = g_n \tilde{G}_n$ with
$$|\tilde{G}_n| < d_1 L^{-\alpha n}, \qquad |\alpha_{n+1} - \alpha_n L^{-\alpha/2}| \leqslant d_2 L^{-\alpha n}$$
3. For $|\phi| > |\log \epsilon|$, $|\mathrm{Im}\,\phi| < L^{-\alpha/2}|\log \epsilon|$,
$$|\tilde{F}_n| \leqslant L^{-\alpha n}(\mathrm{Re}\,\phi)^m \exp\left[-\frac{\bar{\lambda}}{4}\left((\mathrm{Re}\,\phi)^4 + \frac{\kappa}{2}(\mathrm{Re}\,\phi)^2\right)\right]$$

Let again $x_{n_0} = y_{n_0}$, n_0 first such integer. Then

$$\langle \phi_x^m \phi_y^m \rangle_{v_0} = \left\langle \frac{F_{n_0-1}(\psi_{x_{n_0-1}})}{g_{n_0-1}(\psi_{x_{n_0-1}})} \frac{F_{n_0-1}(\psi_{y_{n_0-1}})}{g_{n_0-1}(\psi_{y_{n_0-1}})} \right\rangle_{v_{n_0-1}}$$
$$= L^{-n_0\alpha} \langle \bar{F}_{n_0} g_{n_0}^{-1} \rangle_{v_{n_0}} \tag{67}$$

where $\bar{F}_n$ is now iterated with

1. $\bar{F}_n = \bar{\alpha}_n + \bar{\tilde{F}}_n, \qquad \bar{\tilde{F}}_n(0) = 0$
2. $\bar{\tilde{F}}_n = g_n \bar{\tilde{G}}_n, \quad |\bar{\tilde{G}}_n| < \bar{d}_1 L^{-\alpha/2(n-n_0)}, \quad |\bar{\alpha}_{n+1} - \bar{\alpha}_n| < \bar{d}_2 L^{-\alpha/2(n-n_0)}$
3. $|\bar{\tilde{F}}_n| < \bar{d}_3 (\mathrm{Re}\,\phi)^{2m} L^{-\alpha/2(n-n_0)} \exp\left[-\frac{\bar{\lambda}}{4}\left((\mathrm{Re}\,\phi)^4 + \frac{\kappa}{2}(\mathrm{Re}\,\phi)^2\right)\right]$

The iterations as well as the even case are left as an exercise for the reader. We should note that of course an arbitrarily accurate ($O(\epsilon^2)$) analysis is possible here too.

3. THE LARGE N CASE

The discussion of the multicomponent model parallels the one of the ϵ case; we will hence be brief. In Ref. 1 it was shown that the iteration (1.6), (1.7), with $\alpha = d-2$ ($=1$ for $d=3$, which will be considered here), N large, has a fixed point $g^*(\phi^2)$, conveniently written as

$$g^*(\phi^2) = e^{-(N/2)a^*y}\tilde{g}^*(\phi_0^2 + y) \tag{1}$$

where

$$y = \frac{\phi^2}{N} - \phi_0^2 \equiv \phi^2 - \phi_0^2 \tag{2}$$

$$\phi_0^2 = \frac{L}{L-1} \tag{3}$$

$\tilde{g}^*$ and a^* satisfy the properties (a)–(c) below. The critical manifold in the vicinity of g^* was characterized by the following properties of g_n, each g_n being the RG transformation of g_{n-1}, g_n being attracted to g^*. Fix $\epsilon < \epsilon_0(L)$, $N > N_0(\epsilon, L)$. Then

(a) $g_n(\phi^2) = \tilde{g}_n(\phi_0^2 + y)e^{-(N/2)a_n y}$ with $\tilde{g}_n$ analytic in $|\mathrm{Im}\,\phi| < \epsilon/2\phi_0$, $\tilde{g}_n(\phi_0^2) = 1$, $\tilde{g}_n'(\phi_0^2) = 0$.

(b) For $|y| < \epsilon$,

$$\tilde{g}_n(\phi_0^2 + y) = \exp\left[-\frac{N}{2}\tilde{w}_n(y)\right]$$

with $\tilde{w}_n$ analytic, $\tilde{w}_n(y) = \frac{1}{2}\lambda_n y^2 + \tilde{\tilde{w}}_n(y)$ and

$$|\lambda_n - \lambda^*| \leqslant 3L^{-n/2}\epsilon^{3/2}, \qquad \left|\lambda^* - \frac{L-1}{L^3}\right| \leqslant \epsilon^{3/2}$$

$$|a_n - a^*| \leqslant 3L^{-n/2}\epsilon^2, \qquad |a_n| \leqslant 3\epsilon^2$$

$$|\tilde{\tilde{w}}^{*\prime\prime} - \tilde{\tilde{w}}_n''| \leqslant 3L^{-n/2}\epsilon^{3/4}, \qquad |\tilde{\tilde{w}}_n''| \leqslant \epsilon^{3/4}$$

(c) For $|\mathrm{Im}\,\phi| < L^{-1/2}\epsilon/2\phi_0$, $|y| \geqslant \epsilon$,

$$|\tilde{g}_n(\phi_0^2 + y)| \leqslant \exp\left[-\frac{N}{11}\lambda_\infty(\mathrm{Re}\,y)^2 + \frac{N}{2}\lambda_\infty(\mathrm{Im}\,y)^2\right]$$

with $\lambda_\infty = (L-1)/L^3$.

Let us consider the two-point function. We get

$$G_{xy} = \frac{1}{N}\langle \boldsymbol{\phi}_x \cdot \boldsymbol{\phi}_y \rangle_{v_0} = \frac{1}{N} L^{-(n_0-1)} \langle \boldsymbol{\phi}_{x_{n_0-1}} \cdot \boldsymbol{\phi}_{y_{n_0-1}} \rangle_{v_{n_0-1}}$$
$$= \frac{1}{N} L^{-n_0} \langle \boldsymbol{\phi}_{x_{n_0}} \cdot \boldsymbol{\phi}_{y_{n_0}} \rangle_{v_{n_0}} + \frac{1}{N} L^{-(n_0-1)} \mathsf{A}_{x_{n_0-1}} \mathsf{A}_{y_{n_0-1}} \langle \langle \mathbf{z}^2 \rangle_{n_0-1} \rangle_{v_{n_0}}$$
$$= \sum_{l=n_0-1}^{M-1} L^{-l} \mathsf{A}_l G_M^l(0) \tag{4}$$

as in Section 2. Here

$$G_{l+1}^l = \frac{1}{N}\langle \mathbf{z}^2 \rangle_l, \qquad G_m^l = \langle G_{m-1}^l \rangle_{m-1} \tag{5}$$

Consider computing a z expectation $\langle F(\boldsymbol{\phi}, \mathbf{z}) \rangle_l(\boldsymbol{\phi})$. For an $O(N)$ invariant one (as in $\langle \mathbf{z}^2 \rangle_l$) we may choose coordinates such that $\boldsymbol{\phi} = N^{1/2}(\phi, \mathbf{0})$, $\mathbf{z} = N^{1/2}(s, \xi)$, $\xi^2 \equiv u$ (see Ref. 1) and write

$$\langle F(\boldsymbol{\phi}, \mathbf{z}) \rangle_l = \int_0^\infty du \int_{-\infty}^\infty ds\, F(\boldsymbol{\phi}, \mathbf{z}) G_l(s, u, \phi) \Big/ \int_0^\infty du \int_{-\infty}^\infty ds\, G_l(s, u, \phi) \tag{6}$$

with

$$G_l(s, u, \phi) = \left[\prod_{j=\pm 1} g_l\left(L^{-1}\phi^2 + 2j \frac{\phi s}{L^{1/2}} + s^2 + u \right) \right]^{L^3/2}$$
$$\times \exp\left[-\frac{N}{2}(s^2 + u) + \frac{N-3}{2} \log u \right] \tag{7}$$

Thus

$$G_{l+1}^l = \langle s^2 + u \rangle_l \tag{8}$$

where (with abuse of notation) $\langle - \rangle_l$ now refers to the expectation (6).

Let us analyze $\langle s^2 + u \rangle_l$ now, and take first the case $|y| < \epsilon$. Recall from Ref. 1, that G_l has a stationary point in $s = 0$, $u = u_0(y) = 1 + [(L-1)/L^2]y + O(\epsilon^{7/4})$ in $\mathbb{C}^2$. We deform the s, u contour $\mathbb{R} \times \mathbb{R}^+ \subset \mathbb{C}^2$ to

$$\mathsf{C} = \left[\left(\mathbb{R} \setminus \left[-\frac{\epsilon}{2}, \frac{\epsilon}{2} \right] \right) \times \mathbb{R}^+ \right] \cup \left[-\frac{\epsilon}{2}, \frac{\epsilon}{2} \right] \times C \tag{9}$$

where C is the piecewise linear curve passing through the saddlepoint

$$C = \left[0, 1 - \frac{\epsilon}{2} \right] \cup \left[1 - \frac{\epsilon}{2}, u_0 - \frac{\epsilon}{4} \right] \cup \left[u_0 - \frac{\epsilon}{4}, u_0 + \frac{\epsilon}{4} \right]$$
$$\cup \left[u_0 + \frac{\epsilon}{4}, 1 + \frac{\epsilon}{2} \right] \cup [1 + \epsilon/2, +\infty)$$

(see Fig. 1).

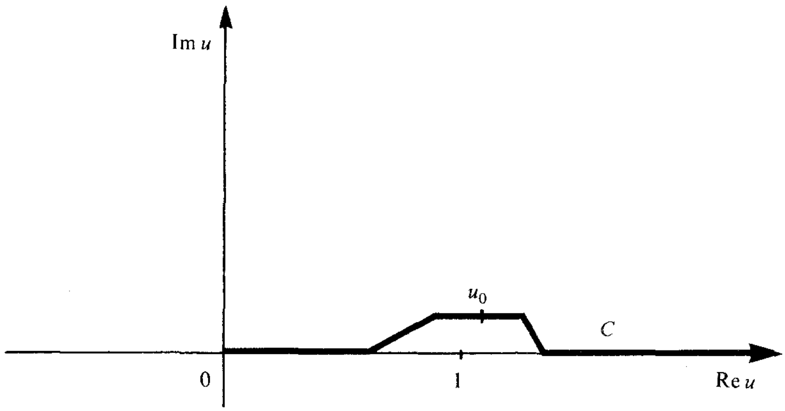

Fig. 1.

Taking

$$C' = \left[-\frac{\epsilon}{2}, \frac{\epsilon}{2} \right] \times \left[u_0 - \frac{\epsilon}{4}, u_0 + \frac{\epsilon}{4} \right]$$

we write

$$\langle F(s,u) \rangle = \langle F(s,u) \rangle_{C'} + \int_{C \setminus C'} FG \Big/ \int_C G$$
$$- \left[\int_{C \setminus C'} G \Big/ \int_C G \right] \langle F(s,u) \rangle_{C'}$$

where $\langle - \rangle_{C'}$ is (6) with $\int_{C'} ds\, du$. Note that we have distorted the contour only in the analyticity region of G (and by assumption, of F); see Ref. 1 where an analogous region was used. Analyze now the (main) term $\langle F \rangle_{C'}$. We need the following properties of G (see Ref. 1). Write $u = u_0(y) + \tilde{u}$. Then

$$G_l(s, u_0 + \tilde{u}, y) = \exp\left[-\frac{N}{2} V_l(s, u_0 + \tilde{u}, y) \right] \tag{10}$$

with

$$V_l(s, u_0 + \tilde{u}, y) = (\sigma, A_l(y)\sigma) + \tilde{V}_l(y, \sigma) + V_l(0, u_0, y) \tag{11}$$

where $\sigma = (s, \tilde{u})$, $A_l(y) = (\partial^2 V / \partial \sigma^2)|_{\sigma=0}$. From (a)–(c) it follows, that (see Ref. 1)

$$A_l(y) = \begin{pmatrix} 6 & 0 \\ 0 & L \end{pmatrix} + O(\epsilon^{3/4}) \tag{12}$$

and

$$|\tilde{V}_l(y,\sigma)| \leqslant C_\epsilon |\sigma|^3 \tag{13}$$

with C_ϵ, e.g., $C\epsilon^{-1/4}$. All the functions are analytic in y.

Make now a change of variables $\sigma' = \sqrt{N}\,\sigma$

$$\langle F(s,u)\rangle_{C'} = \int_{s \leqslant N^{1/2}\epsilon/2} ds \int_{|\tilde{u}| \leqslant N^{1/2}\epsilon/4} d\tilde{u}$$

$$\times F\big(sN^{-1/2}, u_0 + N^{-1/2}\tilde{u}\big)e^{-W(y,\sigma')}/(F=1) \tag{14}$$

with

$$W(y,\sigma') = \tfrac{1}{2}(\sigma', A\sigma') + \tfrac{1}{2}N\tilde{V}(y, N^{-1/2}\sigma') \tag{15}$$

Since $|N\tilde{V}(y,N^{-1/2}\sigma')| \leqslant C_\epsilon N^{-1/2}|\sigma'|^3$, (14) is an expectation in a weakly perturbed (cutoff) Gaussian. Thus easily [denote (14) by $\langle - \rangle_W$] (take also $N^{1/2}\epsilon \gg 1$)

$$|\langle \sigma^n \rangle_w| \leqslant CN^{-n/2} \tag{16}$$

In particular,

$$G'^{l}_{l+1}(y) \equiv \langle s^2 + u \rangle_{C'} = u_{0l}(y) + \bar{G}^{l}_{l+1}(y) \tag{17}$$

with

$$|\bar{G}^{l}_{l+1}(y)| \leqslant \frac{C}{\sqrt{N}} \tag{18}$$

Since $|u_{0l}(y) - u_{0l}(0)| < C|y|$ (see Ref. 1) and

$$\left| \frac{\partial \bar{G}^{l}_{l+1}(y)}{\partial y} \right| \leqslant \frac{C}{\epsilon\sqrt{N}}$$

by analyticity.

$$G'^{l}_{l+n}(y) = G'^{l}_{l+n}(0) + \tilde{G}'^{l}_{l+1}(y) \tag{19}$$

with

$$|\tilde{G}'^{l}_{l+1}(y)| \leqslant C|y| \tag{20}$$

The rest of the contributions to G^{l}_{l+1} in (9) are bounded using the positivity properties of G for large s or u, exactly as in Ref. 1, yielding an $O(e^{-\delta N})$ bound. Thus, for $|y| < \epsilon$,

$$G^{l}_{l+1}(y) = G^{l}_{l+1}(0) + \tilde{G}^{l}_{l+1}(y) \tag{21}$$

with $\tilde{G}^l_{l+1}$ satisfying (20) and

$$G^l_{l+1}(0) = u_{0l}(0) + O\left(\frac{1}{\sqrt{N}}\right) \tag{22}$$

We still need the estimates for $|y| > \epsilon$, $|\mathrm{Im}\,\phi| < \epsilon/2\phi_0\sqrt{L}$. Consider $F^l_{l+1} = g_{l+1}G^l_{l+1}$ given by

$$F^l_{l+1} = \int (s^2 + u)\,G_l(s,u,\phi)\,ds\,du \Big/ \int G_l(s,u,\phi_0)\,ds\,du \tag{23}$$

We bound F^l_{l+1} as we bounded g_{l+1} in Ref. 1, (5.46), (5.47), (5.54)–(5.56). Namely, by (5.46)–(5.47)

$$\begin{aligned}
&\left|\int s^a u^b G_l(s,u,\phi)\,ds\,du\right| \\
&\quad\leqslant \left[\int_0^{3/2} du \int ds \exp\left[-\frac{N}{6}\left(u - 1 + L^2\lambda_\infty(2 + L^3\lambda_\infty)^{-1}\mathrm{Re}\,y\right)^2\right.\right. \\
&\qquad\left.\left. - \frac{N}{3}s^2\right]|s|^a u^b\right] \\
&\quad\times \exp\left[-\frac{N}{6}L(2 + L^3\lambda_\infty)^{-1}\lambda_\infty(\mathrm{Re}\,y)^2 + \frac{N}{2}L\lambda_\infty(\mathrm{Im}\,y)^2\right. \\
&\qquad\left. - \frac{N}{2} + NO(\epsilon^{11/4})\right]\left(1 + O(e^{-\delta N})\right)
\end{aligned} \tag{24}$$

and by (5.55)–(5.57)

$$\begin{aligned}
(24) \leqslant{}& \frac{C}{N^{a/2}}\left(1 + O(N^{-1/2})\right)e^{-\alpha N\epsilon^2}\exp\left[-\frac{N}{11}\lambda_\infty(\mathrm{Re}\,y)^2 + \frac{N}{2}\lambda_\infty(\mathrm{Im}\,y)^2\right] \\
&\times \left|\int ds\,du\,G(s,u,\phi_0)\right|
\end{aligned} \tag{25}$$

(we have extracted an extra $e^{-\alpha N\epsilon^2}$ piece from the marginal term for later convenience).

Thus, again defining

$$F^l_{l+1} = g_{l+1}G^l_{l+1}(0) + \tilde{F}^l_{l+1} \tag{26}$$

we get

$$|\tilde{F}^l_{l+1}| \leqslant 2\exp\left[-\frac{N}{11}\lambda_\infty(\mathrm{Re}\,y)^2 + \frac{N}{2}\lambda_\infty(\mathrm{Im}\,y)^2\right] \tag{27}$$

Iteration of G^l_m now is straightforward. We write

$$G^l_m = \alpha^l_m + \tilde{G}^l_m, \qquad G^l_m(0) = 0 \tag{28}$$

$$F^l_m \equiv g_m G^l_m = \alpha^l_m g_m + \tilde{F}^l_m \tag{29}$$

and prove

$$|\tilde{G}_m^l| \leqslant CL^{-(m-l-1)/2}|y| \tag{30}$$

$$|\alpha_{m+1}^l - \alpha_m^l| \leqslant CL^{-(m-l-1)/2}\epsilon^3 \tag{31}$$

$$|\tilde{F}_m^l| \leqslant 2L^{-(m-l-1)/2}\exp\left[-\frac{N}{11}\lambda_\infty(\operatorname{Re} y)^2 + \frac{N}{2}\lambda_\infty(\operatorname{Im} y)^2\right] \tag{32}$$

The iteration starts from

$$\tilde{F}_{m+1}^l = -(\alpha_{m+1}^l - \alpha_m^l)\, g_{m+1} + \int \tilde{F}_m^l(\tilde{y}) G_m^-(s,u,\phi) \Big/ \int G(s,u,\phi_0) \tag{33}$$

where $\tilde{y} = L^{-1}y + 2\phi s/\sqrt{L} + s^2 + u - 1$ and G_m^- has in (7) one power less of $g_l(L^{-1}\phi^2 + 2\phi s/\sqrt{L} + s^2 + u)$. Assuming (31) and (32), (32) for $m+1$ follows: from the analogs of (24) and (25) we get a smaller multiplier $e^{-\alpha N\epsilon^2} < L^{-1}$, say, to the estimate of the second term of (33).

For (31) note that

$$\alpha_{m+1}^l - \alpha_m^l = \langle \tilde{G}_m^l \rangle(0) = \left\langle \tilde{G}_m^l\left(\frac{2\phi_0 s'}{(LN)^{1/2}} + \frac{s'^2}{N} + \frac{\tilde{u}'}{\sqrt{N}} + u_0(0) - 1\right)\right\rangle_{W(0,\sigma')} + O(e^{-\delta N_L - (m-l)/2}) \tag{34}$$

the first term being bounded by (30), (15), and (12) by

$$C\left[1 + O(\epsilon^{3/4})\right]L^{-(m-l-1)/2}\left[|u_0(0) - 1| + O\left(\frac{1}{\sqrt{N}}\right)\right] \leqslant \frac{1}{2}CL^{-(m-l)}\epsilon^3 \tag{35}$$

since by Ref. 1, Lemma 1, $u_0(0) - 1 = O(\epsilon^{15/4})$. Finally for (30), we get from (33) and (34)

$$\tilde{G}_{m+1}^l = \langle \tilde{G}_m^l(\tilde{y}) \rangle_{W(y,\sigma')} - (y = 0) + \tilde{\tilde{G}}_{m+1}^l - \tilde{\tilde{G}}_{m+1}^l(0) \tag{36}$$

where

$$|\tilde{\tilde{G}}_{m+1}^l| \leqslant O(e^{-\delta N_L - (m-l)/2}) \tag{37}$$

and all terms are analytic. The first two terms in (36) are

$$\int_0^y dy' \frac{\partial}{\partial y'} \left\langle \tilde{G}_m^l\left(L^{-1}y' + \frac{2\phi(y')s'}{(LN)^{1/2}} + \frac{s'^2}{N} + \frac{u'}{\sqrt{N}} + u_0(y') - 1\right)\right\rangle_{W(y,\sigma')} \tag{38}$$

The expectation in (38) is bounded as in (35), using $u_0(y') = 1 - (1/L -$

$1/L^2)y' + O(\epsilon^{3/2})$, by (for $|y'| < 2\epsilon$, say)

$$3L^{-2}\epsilon \cdot CL^{-(m-l-1)/2} \tag{39}$$

and by Cauchy's formula, for $|y'| < \epsilon$

$$\left| \frac{\partial}{\partial y'} \langle - \rangle \right| \leqslant \frac{1}{2} CL^{-(m-l)/2} \tag{40}$$

(37), (38), and (40) yield (30). Thus

$$G_M^l(0) = \left[1 + O(\epsilon^3) \right] \tag{41}$$

and the two-point function has the familiar decay. Moreover, as l increases, by (b) $G^l(0) = \lim_{M \to \infty} G_M^l(0)$ tend to $G^*(0)$ determined solely by the fixed point (we need to know that $|u_{0l}(y) - u_{0*}(y)| = O(L^{-l/2})$, which is easily deduced from the analysis of Ref. 1). Thus the scaling limit is determined solely by the fixed point.

To get the nontriviality of the scaling limit (e.g., for a four-point function, one has to go to the next order in $1/N$. For example, for an $O(N)$ invariant four-point function with points x_i such as in Section 2, we have

$$\begin{aligned}
&\langle (\boldsymbol{\phi}_{x_1} \cdot \boldsymbol{\phi}_{x_2})(\boldsymbol{\phi}_{x_3} \cdot \boldsymbol{\phi}_{x_4}) \rangle_{v_0} \\
&\quad = L^{-2n} \langle (\boldsymbol{\phi}^2_{x_{1n}})^2 \rangle_{v_n} \\
&\qquad + L^{-2n+1} \Big[\langle \boldsymbol{\phi}^2_{x_{1n}} \langle \mathbf{z}^2 \rangle_{n-1} \rangle_{v_n} (\mathsf{A}_{x_{1n-1}} \mathsf{A}_{x_{2n-1}} + \mathsf{A}_{x_{3n-1}} \mathsf{A}_{x_{4n-1}}) \\
&\qquad\qquad + \langle \phi^\alpha_{x_{1n}} \phi^\beta_{x_{2n}} \langle z^\alpha z^\beta \rangle_{n-1} \rangle_{v_n} (\mathsf{A}_{x_{1n-1}} \mathsf{A}_{x_{3n-1}} + \mathsf{A}_{s_{2n-1}} \mathsf{A}_{x_{4n-1}}) \Big] \\
&\qquad + L^{-2(n-1)} \langle \langle (\mathbf{z}^2)^2 \rangle_{n-1} \rangle_{v_n}
\end{aligned} \tag{42}$$

By $O(N)$ invariance again

$$\phi^\alpha \phi^\beta \langle z^\alpha z^\beta \rangle_{n-1} = N^2 \phi^2 \langle s^2 \rangle \tag{43}$$

The non-Gaussian contributions come, e.g., in $\langle (\mathbf{z}^2)^2 \rangle = N^2 \langle (s^2 + u)^2 \rangle$ from the term $\langle \tilde{u}^2 \rangle$, not present in the Gaussian $\langle s^2 + u \rangle \langle s^2 + u \rangle$, which is $O(1/N)$ as well as in the iteration of $\langle u_0(y)^2 \rangle$. Derivation of such systematic $1/N$ expansion is straightforward, albeit tedious and will not be pursued here.

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